Review of Statistics

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Section 1

Introduction

Acknowledgement

Acknowledgement: This chapter is largely based on chapter 3 of "Introduction to Econometrics with R".
https://www.econometrics-with-r.org/index.html

Introduction

The goal of this chapter is

1. Review of Estimation
   - Properties of Estimators: Unbiasedness, Consistency
   - Law of large numbers
2. Review of Central Limit Theorem
   - Important tool for hypothesis testing (to be covered later)

Section 2

Statistical Estimation

Estimation

- Estimator: A mapping from the sample data drawn from an unknown population to a certain feature in the population
  - Example: Consider hourly earnings of college graduates Y .
- You want to estimate the mean of Y, defined as $E[Y] = \mu_y$
  - Draw a random sample of n i.i.d. (identically and independently distributed) observations $Y_1, Y_2, \ldots, Y_N$
- How to estimate $E[Y]$ from the data?

- Idea 1: Sample mean

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i,$$

- Idea 2: Pick the first observation of the sample.
- Question: How can we say which is better?

Properties of the estimator

Consider the estimator $\hat{\mu}_N$ for the unknown parameter μ.

1. Unbiasdeness: The expectation of the estimator is the same as the true parameter in the population.

$$E[\hat{\mu}_N] = \mu$$

2. Consistency: The estimator converges to the true parameter in probability.

$$\forall \epsilon > 0, \lim_{N \to \infty} Prob(|\hat{\mu}_N - \mu| < \epsilon) = 1$$

- Intuition: As the sample size gets larger, the estimator and the true parameter is close with probability one.
- Note: a bit different from the usual convergence of the sequence.

Sample mean $\bar{Y}$ is unbiased and consistent

- Showing these two properties using mathmaetics is straightforward:
  - Unbiasedness: Take expectation.
  - Consistency: Law of large numbers.
- Let's examine these two properties using R programming!

Step 0: Preparing packages

```r
# Use the following packages
library("readr")
library("ggplot2")
library("reshape")

# If not yet, please install by install.packages("").
```

Step 1: Prepare a population

- Use income and age data from PUMS 5% sample of U.S. Census 2000.
    - PUMS: Public Use Microdata Sample
    - Download the example data. Put this file in the same folder as your R script file.
    - https://yutatoyama.github.io/AppliedEconometrics2020/03_Stat/data_pums_2000.csv

```r
# Use "readr" package
library(readr)
pums2000 <- read_csv("data_pums_2000.csv")
```

```
## Parsed with column specification:
## cols(
##   AGE = col_double(),
##   INCTOT = col_double()
## )
```

- We treat this dataset as **population**.

```r
pop <- as.vector(pums2000$INCTOT)
```

- *Population* mean and standard deviation

```r
pop_mean = mean(pop)
pop_sd   = sd(pop)

# Average income in population
pop_mean
```

```
## [1] 30165.47
```

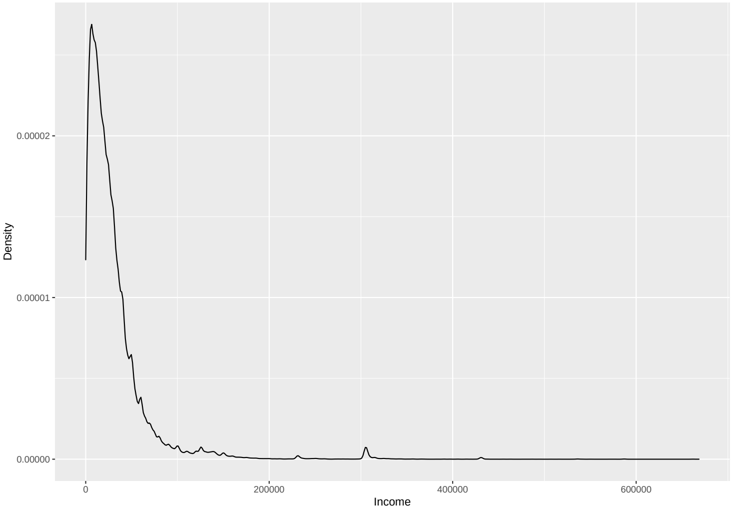
```r
# Standard deviation of income in population
pop_sd
```

```
## [1] 38306.17
```

- income distribution in population (Unit in USD)

```
fig <- ggplot2::qplot(pop, geom = "density",
      xlab = "Income",
      ylab = "Density")
```

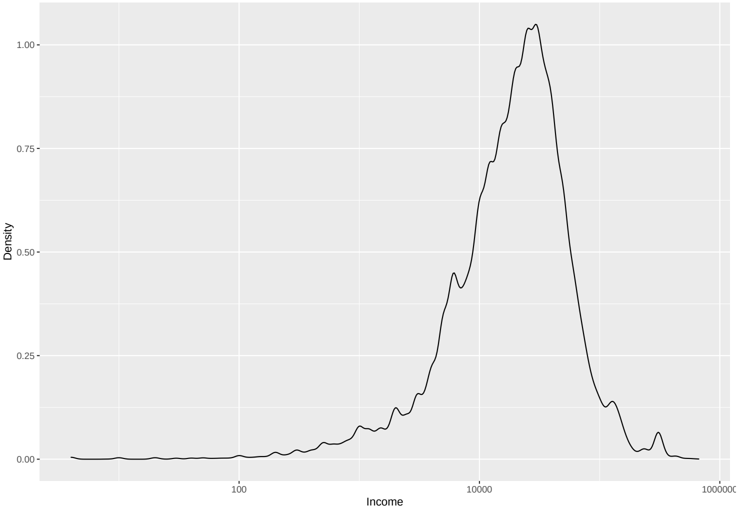
```
plot(fig)
```


0.00002
0.00001
0.00000
Density
0
200000
400000
600000
Income

- The distribution has a long tail.
- Let's plot the distribution in *log* scale

```
# `log` option specifies which axis is represented in log scale.
fig2 <- qplot(pop, geom = "density",
      xlab = "Income",
      ylab = "Density",
      log = "x")
```

```r
plot(fig2)
```



- Let's investigate how close the sample mean constucted from the random sample is to the true population mean.
- Step 1: Draw random samples from this population and calculate $\bar{Y}$ for each sample.
  - Set the sample size N.
- Step 2: Repeat 2000 times. You now have 2000 sample means.

```
# Set the seed for the random number.
# This is needed to maintaine the reproducibility of the results.
set.seed(123)

# draw random sample of 100 observations from the variable pop
test <- sample(x = pop, size = 100)
```

```r
# Use loop to repeat 2000 times.
Nsamples = 2000
result1 <- numeric(Nsamples)

for (i in 1:Nsamples ){

  test <- sample(x = pop, size = 100)
  result1[i] <- mean(test)

}
```

```r
# Anotther way to do this.
result1 <- replicate(expr = mean(sample(x = pop, size = 10)), n = Nsamples)
result2 <- replicate(expr = mean(sample(x = pop, size = 100)),
                     n = Nsamples)
result3 <- replicate(expr = mean(sample(x = pop, size = 500)),
                     n = Nsamples)

# Create dataframe
result_data <- data.frame(  Ybar10 = result1,
                            Ybar100 = result2,
                            Ybar500 = result3)
```

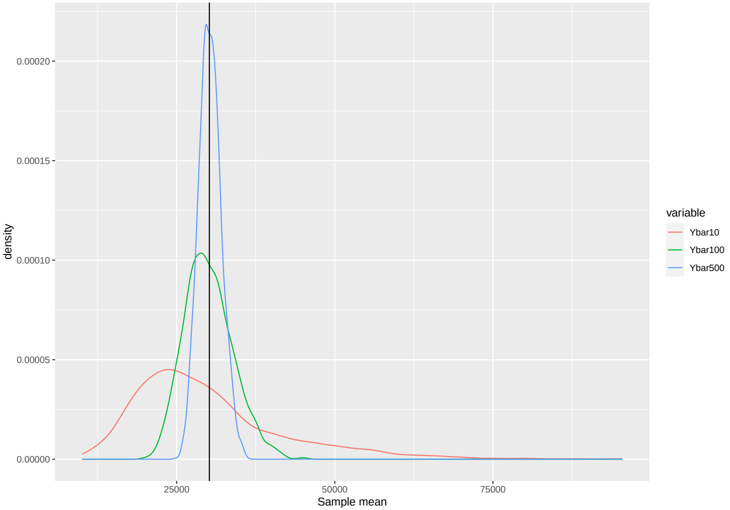
- Step 3: See the distribution of those 2000 sample means.

```
# Use "melt" to change the format of result_data
data_for_plot <- melt(data = result_data, variable.name = "Variable" )
```

```
## Using  as id variables
```

```
# Use "ggplot2" to create the figure.
# The variable `fig` contains the information about the figure
fig <-
  ggplot(data = data_for_plot) +
  xlab("Sample mean") +
  geom_line(aes(x = value, colour = variable ),   stat = "density" ) +
  geom_vline(xintercept=pop_mean ,colour="black")
```

```r
plot(fig)
```


0.00020
0.00015
0.00010
0.00005
0.00000
density
25000
50000
75000
Sample mean
variable
Ybar10
Ybar100
Ybar500

- Observation 1: Regardless of the sample size, the average of the sample means is close to the population mean. **Unbiasdeness**
- Observation 2: As the sample size gets larger, the distribution is concentrated around the population mean. **Consistency (law of large numbers)**

Section 3

Central Limit Theorem

Central limit theorem

- Cental limit theorem: Consider the i.i.d. sample of $Y_1, \cdots, Y_N$ drawn from the random variable Y with mean μ and variance σ^2. The following Z converges in distribution to the normal distribution.

$$Z = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{Y_i - \mu}{\sigma} \xrightarrow{d} N(0,1)$$

In other words,

$$\lim_{N \to \infty} P(Z \leq z) = \Phi(z)$$

What does CLT mean?

- The central limit theorem implies that if N is large **enough**, we can **approximate** the distribution of $\bar{Y}$ by the standard normal distribution with mean μ and variance σ^2/N **regardless of the underlying distribution of Y.**
- This property is called **asymptotic normality**.
- Let's examine this property through simulation!!

Numerical Simulation

- Use the same example as before. Remember that the underlying income distribution is clearly NOT normal.
  - Population mean $\mu = 30165.4673315$
  - standard deviation $\sigma = 38306.1712336$.

```r
# define function for simulation
f_simu_CLT = function(Nsamples, samplesize, pop, pop_mean, pop_sd ){

  output = numeric(Nsamples)
  for (i in 1:Nsamples ){
    test <- sample(x = pop, size = samplesize)
    output[i] <- ( mean(test) - pop_mean ) / (pop_sd / sqrt(samplesize))
  }

  return(output)

}
```

```r
# Set the seed for the random number
set.seed(124)

# Run simulation
Nsamples = 2000
result_CLT1 <- f_simu_CLT(Nsamples, 10, pop, pop_mean, pop_sd )
result_CLT2 <- f_simu_CLT(Nsamples, 100, pop, pop_mean, pop_sd )
result_CLT3 <- f_simu_CLT(Nsamples, 1000, pop, pop_mean, pop_sd )

# Random draw from standard normal distribution as comparison
result_stdnorm = rnorm(Nsamples)

# Create dataframe
result_CLT_data <- data.frame(  Ybar_standardized_10 = result_CLT1,
                            Ybar_standardized_100 = result_CLT2,
                            Ybar_standardized_1000 = result_CLT3,
                            Standard_Normal = result_stdnorm)
```

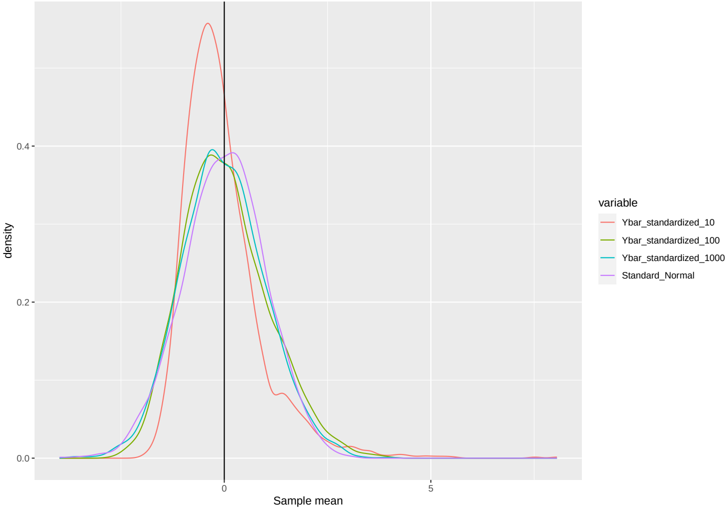
- Now take a look at the distribution.

```
# Use "melt" to change the format of result_data
data_for_plot <- melt(data = result_CLT_data, variable.name = "Variable" )
```

```
## Using  as id variables
```

```
# Use "ggplot2" to create the figure.
fig <-
  ggplot(data = data_for_plot) +
  xlab("Sample mean") +
  geom_line(aes(x = value, colour = variable ),   stat = "density" ) +
  geom_vline(xintercept=0 ,colour="black")
```

```
plot(fig)
```



- As N grows, the distribution is getting closer to the standard normal distribution.